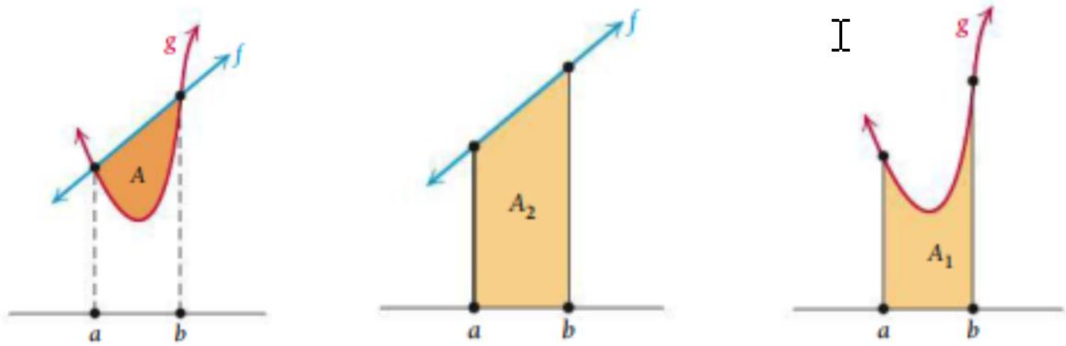
Recall: Finding the Area Bound by Multiple Graphs

$$\int_a^b \left[f(x) - g(x) \right] dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$



Step 1: determine which function has larger y-values

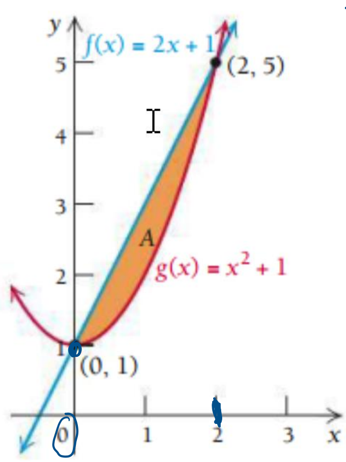
Step 2: determine integration bounds by finding the functions' intersection points (set functions equal to each other)

recall: find the area of the region **bound** by the graphs $\underbrace{f(x) = 2x+1}_{}$, $g(x) = x^2 + 1$

Y_{TOP} (above f), Y_{BOTTOM} (above g)

get intersection points (bounds)

$$2x+1 = x^2+1$$
$$0 = x^2 - 2x$$
$$0 = x(x-2)$$
$$x = 0 \qquad x = 2$$

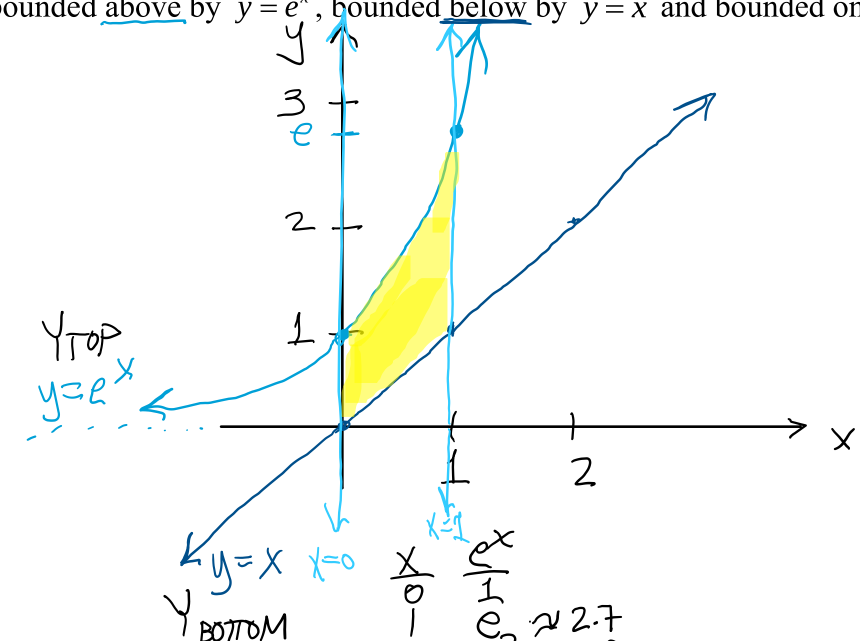


$$A = \int_0^2 (Y_{TOP} - Y_{BOTTOM})\, dx$$
$$= \int_0^2 \left(2x+1 - (x^2+1)\right) dx$$
$$= \int_0^2 \left(2x + \cancel{1} - x^2 - \cancel{1}\right) dx$$
$$= \left[x^2 - \frac{x^3}{3} \right]_0^2$$
$$= \frac{3}{3} \cdot 4 - \frac{8}{3} = \frac{12-8}{3} = \boxed{\frac{4}{3}}$$

ex. Use an integral to find area of a region bounded above by $y = e^x$, bounded below by $y = x$ and bounded on the sides by vertical lines $x = 0$ and $x = 1$.

don't intersect :)

$$A = \int_0^1 (y_T - y_B)\,dx$$
$$= \int_0^1 (e^x - x)\,dx$$
$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$
$$= \underbrace{e^1 - \frac{1}{2}}_{f(b)} - \underbrace{(e^0 - 0)}_{f(a)}$$
$$= e - \frac{1}{2} - 1$$
$$= \boxed{e - \frac{3}{2}}$$



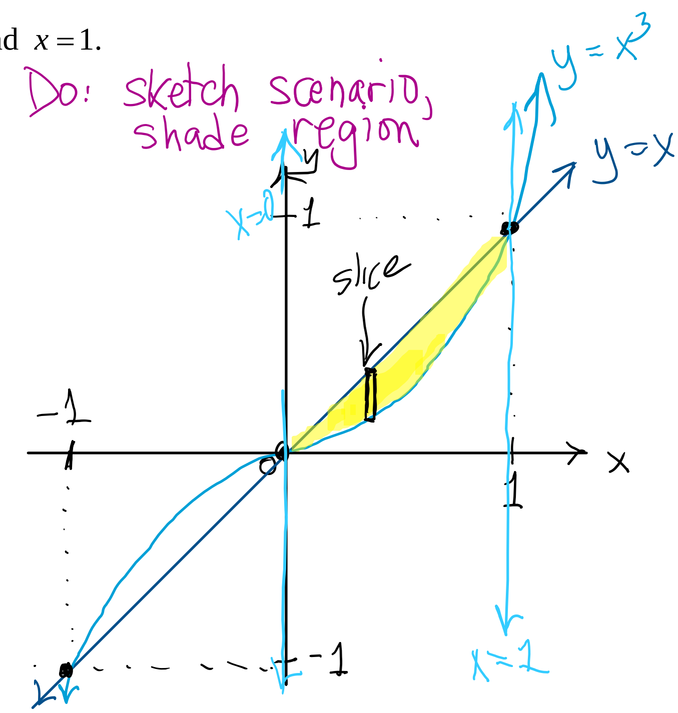
x	e^x
0	1
1	$e \approx 2.7$
2	$e^2 \approx 7.3$

ex. Find area of the region bounded by $y = x$, $y = x^3$, $x = 0$ and $x = 1$.

Do: intersection pts

$$x = x^3$$
$$0 = x^3 - x$$
$$0 = x(x^2 - 1)$$
$$= x(x+1)(x-1)$$
$$x = 0 \quad x = -1 \quad x = 1$$

Do: sketch scenario, shade region



Do: set up and solve integral

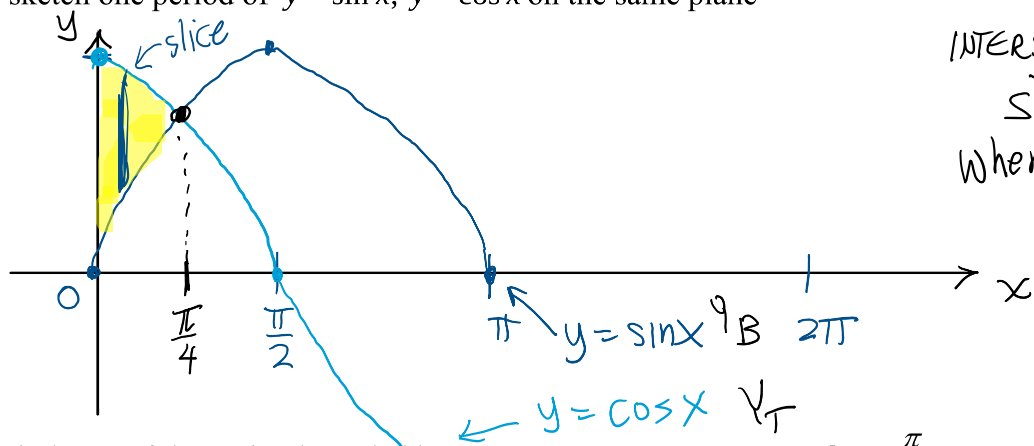
$$A = \int_0^1 (y_T - y_B)\,dx$$
$$= \int_0^1 (x - x^3)\,dx$$
$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$
$$= \frac{2}{2}\cdot\frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$$

$$\int_{-1}^{1} (x - x^3)\,dx = \boxed{0}$$

$f(x) = x - x^3 \leftarrow$ odd

$$f(-x) = -x - (-x)^3$$
$$= -x - (-x^3)$$
$$= -x + x^3 = -(x - x^3) = -f(x)$$

Do: sketch one period of $y = \sin x,\ y = \cos x$ on the same plane



INTERSECTION POINTS:

$\sin x = \cos x$

when $x = \frac{\pi}{4}$

ex. Find area of the region bounded by $y = \sin x,\ y = \cos x,\ x = 0$ and $x = \frac{\pi}{4}$.

$$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

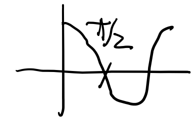
$$= \Big[\sin x + \cos x\Big]_0^{\pi/4}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1$$

$$= \frac{\cancel{2}\sqrt{2}}{\cancel{2}} - 1$$

$$= \boxed{\sqrt{2} - 1}$$



ex. Find area of the region bounded by $y = \sin 2x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

INTERSECTION POINTS:

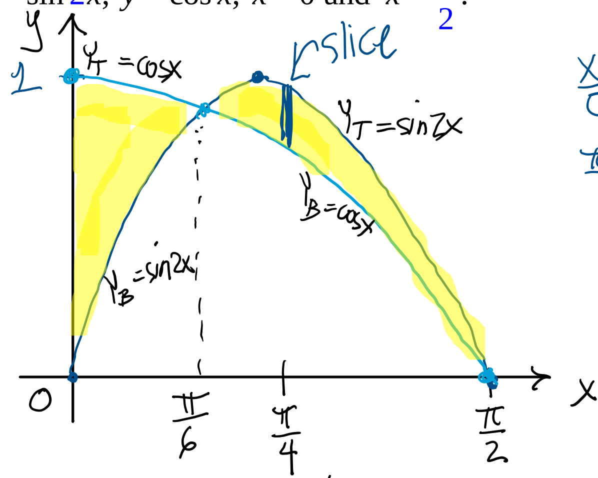
$\sin 2x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ OR $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ $\quad$ $x = \frac{\pi}{6}$

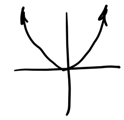


x	$\sin 2x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\sin\left(2\cdot\frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \boxed{\frac{\sqrt{3}}{2}}$	$\boxed{\sqrt{3}/2}$
$\frac{\pi}{2}$	$\sin 2\cdot\frac{\pi}{2} = \sin\pi = \boxed{0}$	$\stackrel{?}{=}$ $\boxed{0}$
$\frac{\pi}{4}$	$\sin\frac{\pi}{2} = 1$	

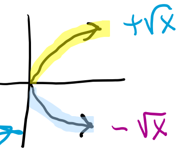
use 2 integrals

$$A = \int_0^{\pi/6}(\cos x - \sin 2x)\,dx + \int_{\pi/6}^{\pi/2}(\sin 2x - \cos x)\,dx = \boxed{\frac{1}{2}}$$

Sometimes it makes more sense to integrate with respect to *y*: $y=x^2$



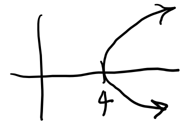
$x=y^2$



$\sqrt{x}=\sqrt{y^2}$

$\pm\sqrt{x}=y$

ex. Find area enclosed by $x=2y^2$ and $x=4+y^2$.



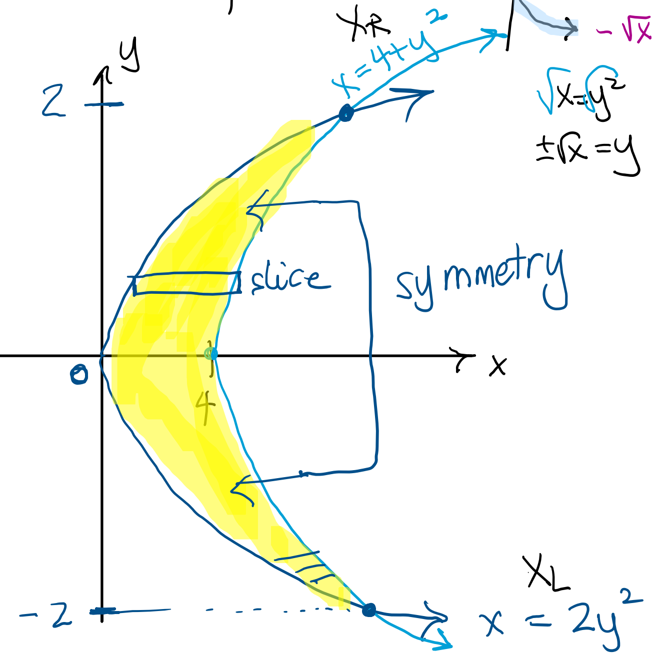
INTERSECTION POINTS

$2y^2=4+y^2$ solve for y

$y^2-4=0$

$(y+2)(y-2)=0$

$y=-2 \quad y=2$



$A=\int_{-2}^{2}(x_{RIGHT}-x_{LEFT})\,dy$

$=\int_{-2}^{2}(4+y^2-2y^2)\,dy$

$=2\int_0^2(4-y^2)\,dy$

$=2\left(4y-\frac{y^3}{3}\right)\Big|_0^2$

$=2\left(\frac{3}{3}8-\frac{8}{3}\right)=2\cdot\frac{24-8}{3}=\frac{2\cdot16}{3}=\boxed{\frac{32}{3}}$

ex. Find area enclosed by $y=x-1$ and $y^2=2x+6$.

write both ITO x

$x=y+1 \qquad \frac{y^2}{2}-\frac{6}{2}=\frac{2x}{2}$

$\frac{1}{2}y^2-3=x$

INTERSECTION POINTS:

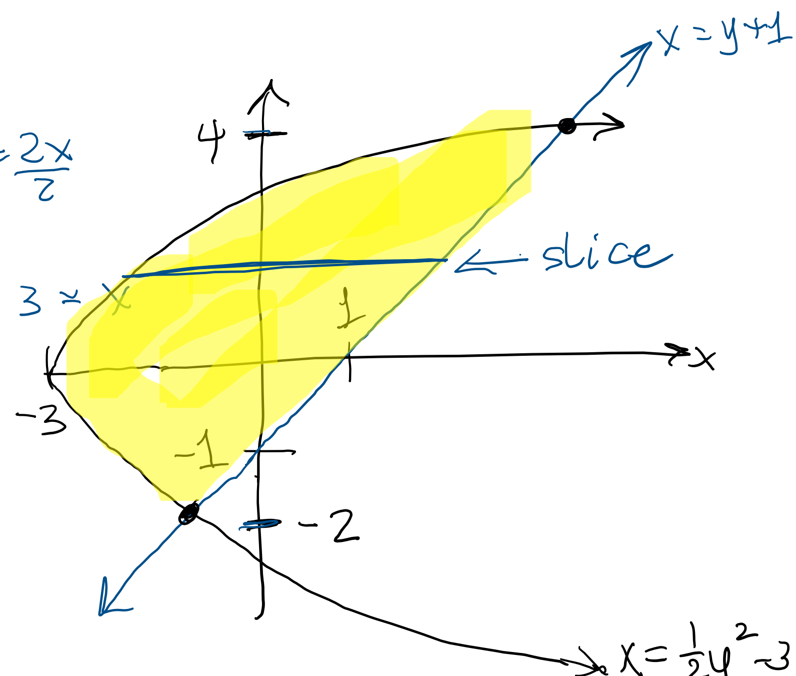
$2\left(y+1=\frac{1}{2}y^2-3\right)$

$2y+2=y^2-6$

$0=y^2-2y-8$

$0=(y+2)(y-4)$

$y=-2 \quad y=4$



$A=\int_{-2}^{4}(x_R-x_L)\,dy$

$=\int_{-2}^{4}\left(y+1-\left(\frac{1}{2}y^2-3\right)\right)dy$

combine like terms

integrate

evaluate $\rightarrow \boxed{18}$